

Two linear transformations each tridiagonal with respect to an eigenbasis of the other; comments on the split decomposition^{*}

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Dedicated to Tom Koornwinder on his 60th birthday

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Abstract

Let \mathbb{K} denote a field and let d denote a nonnegative integer. Let \mathcal{A} denote a \mathbb{K} -algebra isomorphic to $\text{Mat}_{d+1}(\mathbb{K})$. An element of \mathcal{A} is called *multiplicity-free* whenever its eigenvalues are mutually distinct and contained in \mathbb{K} . Let A and A^* denote multiplicity-free elements in \mathcal{A} . Let $\{E_i\}_{i=0}^d$ (resp. $\{E_i^*\}_{i=0}^d$) denote an ordering of the primitive idempotents of A (resp. A^* .) For $0 \leq i \leq d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) for E_i (resp. E_i^* .) Let V denote an irreducible left \mathcal{A} -module. By a *decomposition* of V we mean a sequence $\{U_i\}_{i=0}^d$ consisting of 1-dimensional subspaces of V such that $V = \sum_{i=0}^d U_i$. A decomposition $\{U_i\}_{i=0}^d$ of V is said to be *split* (with respect to the orderings $\{E_i\}_{i=0}^d, \{E_i^*\}_{i=0}^d$) whenever both (i) $(A - \theta_i I)U_i = U_{i+1}$ ($0 \leq i \leq d-1$), $(A - \theta_d I)U_d = 0$; and (ii) $(A^* - \theta_i^* I)U_i = U_{i-1}$ ($1 \leq i \leq d$), $(A^* - \theta_0^* I)U_0 = 0$. We show there exists at most one decomposition of V which is split with respect to $\{E_i\}_{i=0}^d, \{E_i^*\}_{i=0}^d$. We show the following are equivalent: (i) there exists a decomposition of V which is split with respect to $\{E_i\}_{i=0}^d, \{E_i^*\}_{i=0}^d$; (ii) both

$$E_i^* A E_j^* = \begin{cases} 0, & \text{if } i - j > 1; \\ \neq 0, & \text{if } i - j = 1 \end{cases} \quad E_i A^* E_j = \begin{cases} 0, & \text{if } j - i > 1; \\ \neq 0, & \text{if } j - i = 1 \end{cases}$$

for $0 \leq i, j \leq d$. We call the sequence $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ a *Leonard system* whenever both

$$E_i^* A E_j^* = \begin{cases} 0, & \text{if } |i - j| > 1; \\ \neq 0, & \text{if } |i - j| = 1 \end{cases} \quad E_i A^* E_j = \begin{cases} 0, & \text{if } |i - j| > 1; \\ \neq 0, & \text{if } |i - j| = 1 \end{cases}$$

for $0 \leq i, j \leq d$. We show $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system if and only if both (i) there exists a decomposition of V which is split with respect to $\{E_i\}_{i=0}^d, \{E_i^*\}_{i=0}^d$; (ii) there exists a decomposition of V which is split with respect to $\{E_{d-i}\}_{i=0}^d, \{E_i^*\}_{i=0}^d$. We also show $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system if and only if both (i) there exists a decomposition of V which is split with respect to $\{E_i\}_{i=0}^d, \{E_i^*\}_{i=0}^d$; (ii) there exists an antiautomorphism \dagger of \mathcal{A} such that $A^\dagger = A$ and $A^{*\dagger} = A^*$.

1 Leonard pairs and Leonard systems

We begin by recalling the notion of a *Leonard pair* [6], [12], [13], [14], [15], [16], [17], [18]. We will use the following terms. Let X denote a square matrix. Then X is called *tridiagonal* whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. Assume X is tridiagonal. Then X is called *irreducible* whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

We now define a Leonard pair. For the rest of this paper \mathbb{K} will denote a field.

Definition 1.1 [13, Definition 1.1] Let V denote a vector space over \mathbb{K} with finite positive dimension. By a *Leonard pair* on V , we mean an ordered pair of linear transformations $A : V \rightarrow V$ and $A^* : V \rightarrow V$ which satisfy both (i), (ii) below.

- (i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
- (ii) There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

Note 1.2 According to a common notational convention A^* denotes the conjugate-transpose of A . We emphasize we are not using this convention. In a Leonard pair A, A^* the linear transformations A and A^* are arbitrary subject to (i), (ii) above.

Our use of the name “Leonard pair” is motivated by a connection to a theorem of D. Leonard [2, p. 260], [9] which involves the q -Racah polynomials [1], [3, p. 162] and some related polynomials of the Askey scheme [7]. This connection is discussed in [13, Appendix A] and [15, Section 16]. See [4], [5], [8], [10], [19] for related topics.

When working with a Leonard pair, it is often convenient to consider a closely related and somewhat more abstract concept called a *Leonard system*. In order to define this we recall a few more terms. Let d denote a nonnegative integer. Let $\text{Mat}_{d+1}(\mathbb{K})$ denote the \mathbb{K} -algebra consisting of all $d+1$ by $d+1$ matrices which have entries in \mathbb{K} . We index the rows and columns by $0, 1, \dots, d$. Let \mathbb{K}^{d+1} denote the \mathbb{K} -vector space consisting of all $d+1$ by 1 matrices which have entries in \mathbb{K} . We index the rows by $0, 1, \dots, d$. We view \mathbb{K}^{d+1} as a left module for $\text{Mat}_{d+1}(\mathbb{K})$ under matrix multiplication. We observe this module is irreducible. For the rest of this paper we let \mathcal{A} denote a \mathbb{K} -algebra isomorphic to $\text{Mat}_{d+1}(\mathbb{K})$. When we refer to an \mathcal{A} -module we mean a left \mathcal{A} -module. Let V denote an irreducible \mathcal{A} -module. We remark that V is unique up to isomorphism of \mathcal{A} -modules and that V has dimension $d+1$. Let v_0, v_1, \dots, v_d denote a basis for V . For $X \in \mathcal{A}$ and for $Y \in \text{Mat}_{d+1}(\mathbb{K})$, we say Y represents X with respect to v_0, v_1, \dots, v_d whenever $Xv_j = \sum_{i=0}^d Y_{ij}v_i$ for $0 \leq j \leq d$. Let A denote an element of \mathcal{A} . We say A is *multiplicity-free* whenever it has $d+1$ distinct eigenvalues in \mathbb{K} . Assume A is multiplicity-free. Let $\theta_0, \theta_1, \dots, \theta_d$ denote an ordering of the eigenvalues of A , and for $0 \leq i \leq d$ put

$$E_i = \prod_{\substack{0 \leq j \leq d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j},$$

where I denotes the identity of \mathcal{A} . We observe (i) $AE_i = \theta_i E_i$ ($0 \leq i \leq d$); (ii) $E_i E_j = \delta_{ij} E_i$ ($0 \leq i, j \leq d$); (iii) $\sum_{i=0}^d E_i = I$; (iv) $A = \sum_{i=0}^d \theta_i E_i$. Let \mathcal{D} denote the subalgebra of \mathcal{A} generated by A . Using (i)–(iv) we find E_0, E_1, \dots, E_d is a basis for the \mathbb{K} -vector space \mathcal{D} . We call E_i the *primitive idempotent* of A associated with θ_i . It is helpful to think of these primitive idempotents as follows. Observe

$$V = E_0 V + E_1 V + \dots + E_d V \quad (\text{direct sum}).$$

For $0 \leq i \leq d$, $E_i V$ is the (one dimensional) eigenspace of A in V associated with the eigenvalue θ_i , and E_i acts on V as the projection onto this eigenspace. We remark that the sequence $\{A^i | 0 \leq i \leq d\}$ is a basis for the \mathbb{K} -vector space \mathcal{D} and that $\prod_{i=0}^d (A - \theta_i I) = 0$. By a *Leonard pair in \mathcal{A}* we mean an ordered pair of elements taken from \mathcal{A} which act on V as a Leonard pair in the sense of Definition 1.1. We now define a Leonard system.

Definition 1.3 [13, Definition 1.4] By a *Leonard system* in \mathcal{A} , we mean a sequence $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ which satisfies (i)–(v) below.

- (i) Each of A, A^* is a multiplicity-free element of \mathcal{A} .
- (ii) E_0, E_1, \dots, E_d is an ordering of the primitive idempotents of A .
- (iii) $E_0^*, E_1^*, \dots, E_d^*$ is an ordering of the primitive idempotents of A^* .
- (iv) $E_i^* A E_j^* = \begin{cases} 0, & \text{if } |i - j| > 1; \\ \neq 0, & \text{if } |i - j| = 1 \end{cases} \quad (0 \leq i, j \leq d).$
- (v) $E_i A^* E_j = \begin{cases} 0, & \text{if } |i - j| > 1; \\ \neq 0, & \text{if } |i - j| = 1 \end{cases} \quad (0 \leq i, j \leq d).$

We comment on how Leonard pairs and Leonard systems are related. In the following discussion V denotes an irreducible \mathcal{A} -module. Let $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in \mathcal{A} . For $0 \leq i \leq d$ let v_i denote a nonzero vector in $E_i V$. Then the sequence v_0, v_1, \dots, v_d is a basis for V which satisfies Definition 1.1(ii). For $0 \leq i \leq d$ let v_i^* denote a nonzero vector in $E_i^* V$. Then the sequence $v_0^*, v_1^*, \dots, v_d^*$ is a basis for V which satisfies Definition 1.1(i). By these comments the pair A, A^* is a Leonard pair in \mathcal{A} . Conversely let A, A^* denote a Leonard pair in \mathcal{A} . By [13, Lemma 1.3] each of A, A^* is multiplicity-free. Let v_0, v_1, \dots, v_d denote a basis for V which satisfies Definition 1.1(ii). For $0 \leq i \leq d$ the vector v_i is an eigenvector for A ; let E_i denote the corresponding primitive idempotent. Let $v_0^*, v_1^*, \dots, v_d^*$ denote a basis for V which satisfies Definition 1.1(i). For $0 \leq i \leq d$ the vector v_i^* is an eigenvector for A^* ; let E_i^* denote the corresponding primitive idempotent. Then $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system in \mathcal{A} . In summary we have the following.

Lemma 1.4 *Let A and A^* denote elements in \mathcal{A} . Then the pair A, A^* is a Leonard pair in \mathcal{A} if and only if the following (i), (ii) hold.*

- (i) *Each of A, A^* is multiplicity-free.*

- (ii) There exists an ordering E_0, E_1, \dots, E_d of the primitive idempotents of A and there exists an ordering $E_0^*, E_1^*, \dots, E_d^*$ of the primitive idempotents of A^* such that $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system in \mathcal{A} .

Later in this paper we will obtain two characterizations of a Leonard system. These characterizations are based on a concept which we call the *split decomposition*. This concept is explained in the next section.

2 The split decomposition

In [13] we introduced the split decomposition for Leonard systems and in [15] we discussed this decomposition in detail. For our present purposes it is useful to define the split decomposition in a more general context. We will refer to the following set-up.

Definition 2.1 Let A and A^* denote multiplicity-free elements in \mathcal{A} . Let E_0, E_1, \dots, E_d denote an ordering of the primitive idempotents of A and for $0 \leq i \leq d$ let θ_i denote the eigenvalue of A for E_i . Let $E_0^*, E_1^*, \dots, E_d^*$ denote an ordering of the primitive idempotents of A^* and for $0 \leq i \leq d$ let θ_i^* denote the eigenvalue of A^* for E_i^* . We let \mathcal{D} (resp. \mathcal{D}^*) denote the subalgebra of \mathcal{A} generated by A (resp. A^* .) We let V denote an irreducible \mathcal{A} -module.

With reference to Definition 2.1, by a *decomposition* of V we mean a sequence U_0, U_1, \dots, U_d consisting of 1-dimensional subspaces of V such that

$$V = U_0 + U_1 + \dots + U_d \quad (\text{direct sum}).$$

We have a comment. Let u_0, u_1, \dots, u_d denote a basis for V and for $0 \leq i \leq d$ let U_i denote the subspace of V spanned by u_i . Then the sequence U_0, U_1, \dots, U_d is a decomposition of V . Conversely, let U_0, U_1, \dots, U_d denote a decomposition of V . For $0 \leq i \leq d$ let u_i denote a nonzero vector in U_i . Then u_0, u_1, \dots, u_d is a basis for V .

Definition 2.2 With reference to Definition 2.1, let U_0, U_1, \dots, U_d denote a decomposition of V . We say this decomposition is *split* (with respect to the orderings E_0, E_1, \dots, E_d and $E_0^*, E_1^*, \dots, E_d^*$) whenever both

$$(A - \theta_i I)U_i = U_{i+1} \quad (0 \leq i \leq d-1), \quad (A - \theta_d I)U_d = 0, \quad (1)$$

$$(A^* - \theta_i^* I)U_i = U_{i-1} \quad (1 \leq i \leq d), \quad (A^* - \theta_0^* I)U_0 = 0. \quad (2)$$

Later in this paper we will obtain two characterizations of a Leonard system which involve the split decomposition. For the time being we consider the existence and uniqueness of the split decomposition. We start with uniqueness.

Lemma 2.3 *With reference to Definition 2.1, the following (i), (ii) hold.*

- (i) Assume there exists a decomposition U_0, U_1, \dots, U_d of V which is split with respect to the orderings E_0, E_1, \dots, E_d and $E_0^*, E_1^*, \dots, E_d^*$. Then $U_i = \prod_{h=0}^{i-1} (A - \theta_h I) E_0^* V$ and $U_i = \prod_{h=i+1}^d (A^* - \theta_h^* I) E_d V$ for $0 \leq i \leq d$.

(ii) There exists at most one decomposition of V which is split with respect to the orderings E_0, E_1, \dots, E_d and $E_0^*, E_1^*, \dots, E_d^*$.

Proof: (i) From the equation on the right in (2) we find $U_0 = E_0^*V$. Using this and (1) we obtain $U_i = \prod_{h=0}^{i-1} (A - \theta_h I) E_0^*V$ for $0 \leq i \leq d$. From the equation on the right in (1) we find $U_d = E_dV$. Using this and (2) we obtain $U_i = \prod_{h=i+1}^d (A^* - \theta_h^* I) E_dV$ for $0 \leq i \leq d$.
(ii) Immediate from (i) above. \square

We turn our attention to the existence of the split decomposition. In Section 4 we will give a necessary and sufficient condition for this existence. We will use the following result.

Lemma 2.4 *With reference to Definition 2.1, assume there exists a decomposition U_0, U_1, \dots, U_d of V which is split with respect to the orderings E_0, E_1, \dots, E_d and $E_0^*, E_1^*, \dots, E_d^*$. Then the following (i)–(v) hold for $0 \leq i \leq d$.*

- (i) $\sum_{h=0}^i U_h = \sum_{h=0}^i A^h E_0^*V$.
- (ii) $\sum_{h=0}^i U_h = \sum_{h=0}^i E_h^*V$.
- (iii) $\sum_{h=i}^d U_h = \sum_{h=0}^{d-i} A^{*h} E_dV$.
- (iv) $\sum_{h=i}^d U_h = \sum_{h=i}^d E_hV$.
- (v) $U_i = (E_0^*V + E_1^*V + \dots + E_i^*V) \cap (E_iV + E_{i+1}V + \dots + E_dV)$.

Proof: (i) For $0 \leq j \leq d$ we have $U_j = \prod_{h=0}^{j-1} (A - \theta_h I) E_0^*V$ by Lemma 2.3(i) so $U_j \subseteq \sum_{h=0}^j A^h E_0^*V$. Apparently $\sum_{h=0}^i U_h \subseteq \sum_{h=0}^i A^h E_0^*V$. In this inclusion the sum on the left has dimension $i + 1$ since U_0, U_1, \dots, U_d is a decomposition. The sum on the right has dimension at most $i + 1$. Therefore $\sum_{h=0}^i U_h = \sum_{h=0}^i A^h E_0^*V$.
(ii) For $0 \leq j \leq d$ we have $\prod_{h=0}^j (A^* - \theta_h^* I) U_j = 0$ by (2) so $U_j \subseteq \sum_{h=0}^j E_h^*V$. Apparently $\sum_{h=0}^i U_h \subseteq \sum_{h=0}^i E_h^*V$. In this inclusion each side has dimension $i + 1$ so equality holds.
(iii) Similar to the proof of (i) above.
(iv) Similar to the proof of (ii) above.
(v) Combine (ii), (iv) above. \square

3 Some products

Our next goal is to display a necessary and sufficient condition for the existence of the split decomposition. With reference to Definition 2.1, consider the products

$$E_i^* A E_j^*, \quad E_i A^* E_j \quad (0 \leq i, j \leq d).$$

Our condition has to do with which of these products is 0. In order to motivate our result we initially consider just one of these products.

Lemma 3.1 *With reference to Definition 2.1, for $0 \leq i \leq d$ let v_i^* denote a nonzero vector in E_i^*V and observe $v_0^*, v_1^*, \dots, v_d^*$ is a basis for V . Let B denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ which represents A with respect to this basis, so that*

$$Av_j^* = \sum_{i=0}^d B_{ij}v_i^* \quad (0 \leq j \leq d). \quad (3)$$

*Then for $0 \leq i, j \leq d$ the following are equivalent: (i) $E_i^*AE_j^* = 0$; (ii) $B_{ij} = 0$.*

Proof: Let the integers i, j be given. Observe $E_r^*v_s^* = \delta_{rs}v_s^*$ for $0 \leq r, s \leq d$. By this and (3) we find $E_i^*AE_j^*V$ is spanned by $B_{ij}v_i^*$. The result follows. \square

In the next lemma we consider a certain pattern of vanishing products among the $E_i^*AE_j^*$. We will use the following notation. Let λ denote an indeterminate and let $\mathbb{K}[\lambda]$ denote the \mathbb{K} -algebra consisting of all polynomials in λ which have coefficients in \mathbb{K} . Let f_0, f_1, \dots, f_d denote a sequence of polynomials taken from $\mathbb{K}[\lambda]$. We say this sequence is *graded* whenever f_i has degree exactly i for $0 \leq i \leq d$.

Lemma 3.2 *With reference to Definition 2.1, the following (i)–(iii) are equivalent.*

$$(i) \ E_i^*AE_j^* = \begin{cases} 0, & \text{if } i - j > 1; \\ \neq 0, & \text{if } i - j = 1 \end{cases} \quad (0 \leq i, j \leq d).$$

(ii) *There exists a graded sequence of polynomials f_0, f_1, \dots, f_d taken from $\mathbb{K}[\lambda]$ such that $E_i^*V = f_i(A)E_0^*V$ for $0 \leq i \leq d$.*

(iii) *For $0 \leq i \leq d$,*

$$\sum_{h=0}^i E_h^*V = \sum_{h=0}^i A^h E_0^*V. \quad (4)$$

Proof: (i) \Rightarrow (ii) For $0 \leq i \leq d$ let v_i^* denote a nonzero vector in E_i^*V and observe $v_0^*, v_1^*, \dots, v_d^*$ is a basis for V . Let B denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ which represents A with respect to this basis. By Lemma 3.1,

$$B_{ij} = \begin{cases} 0, & \text{if } i - j > 1; \\ \neq 0, & \text{if } i - j = 1 \end{cases} \quad (0 \leq i, j \leq d). \quad (5)$$

Let f_0, f_1, \dots, f_d denote the polynomials in $\mathbb{K}[\lambda]$ which satisfy $f_0 = 1$ and

$$\lambda f_j = \sum_{i=0}^{j+1} B_{ij}f_i \quad (0 \leq j \leq d-1). \quad (6)$$

We observe f_i has degree exactly i for $0 \leq i \leq d$ so the sequence f_0, f_1, \dots, f_d is graded. Comparing (3) and (6) in light of (5) we find $v_i^* = f_i(A)v_0^*$ for $0 \leq i \leq d$. It follows $E_i^*V = f_i(A)E_0^*V$ for $0 \leq i \leq d$.

(ii) \Rightarrow (iii) For $0 \leq j \leq d$ we have $E_j^*V = f_j(A)E_0^*V$. The degree of f_j is j so $E_j^*V \subseteq \sum_{h=0}^j A^h E_0^*V$. Apparently $\sum_{h=0}^i E_h^*V \subseteq \sum_{h=0}^i A^h E_0^*V$. In this inclusion the sum on the left has dimension $i+1$ and the sum on the right has dimension at most $i+1$. Therefore $\sum_{h=0}^i E_h^*V = \sum_{h=0}^i A^h E_0^*V$.

(iii) \Rightarrow (i) For $0 \leq i \leq d$ let V_i denote the subspace on the left or right in (4). From the right-hand side of (4) we find $V_i + AV_i = V_{i+1}$ for $0 \leq i \leq d-1$. From the left-hand side of (4) we find $E_r^*V_s = 0$ for $0 \leq s < r \leq d$. Let i, j denote integers ($0 \leq i, j \leq d$) and first assume $i - j > 1$. We show $E_i^*AE_j^* = 0$. Observe $E_j^*V \subseteq V_j$ and $AV_j \subseteq V_{j+1}$ so $AE_j^*V \subseteq V_{j+1}$. However $E_i^*V_{j+1} = 0$ since $i - j > 1$ so $E_i^*AE_j^*V = 0$. It follows $E_i^*AE_j^* = 0$. Next we assume $i - j = 1$ and show $E_i^*AE_j^* \neq 0$. Suppose $E_i^*AE_j^* = 0$. Then by our previous remarks $E_i^*AE_h^* = 0$ for $0 \leq h \leq j$. By this and since $V_j = \sum_{h=0}^j E_h^*V$ we find $E_i^*AV_j = 0$. However $V_i = V_j + AV_j$ and $E_i^*V_j = 0$ so $E_i^*V_i = 0$. This contradicts the construction so $E_i^*AE_j^* \neq 0$. \square

Corollary 3.3 *With reference to Definition 2.1, let v_0^* denote a nonzero vector in E_0^*V and consider the \mathbb{K} -linear transformation from \mathcal{D} to V which sends X to Xv_0^* for all $X \in \mathcal{D}$. Assume the equivalent conditions (i)–(iii) hold in Lemma 3.2. Then this linear transformation is an isomorphism.*

Proof: Since the \mathbb{K} -vector spaces \mathcal{D} and V have the same dimension it suffices to show the linear transformation is surjective. Setting $i = d$ in (4) we find $V = \mathcal{D}v_0^*$. Therefore the linear transformation is surjective. \square

Replacing $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ by $(A^*; A; \{E_{d-i}^*\}_{i=0}^d; \{E_{d-i}\}_{i=0}^d)$ in Lemma 3.2 and Corollary 3.3 we routinely obtain the following results.

Lemma 3.4 *With reference to Definition 2.1, the following (i)–(iii) are equivalent.*

$$(i) \quad E_i A^* E_j = \begin{cases} 0, & \text{if } j - i > 1; \\ \neq 0, & \text{if } j - i = 1 \end{cases} \quad (0 \leq i, j \leq d).$$

(ii) *There exists a graded sequence of polynomials $f_0^*, f_1^*, \dots, f_d^*$ taken from $\mathbb{K}[\lambda]$ such that $E_i V = f_{d-i}^*(A^*)E_d V$ for $0 \leq i \leq d$.*

(iii) *For $0 \leq i \leq d$,*

$$\sum_{h=i}^d E_h V = \sum_{h=0}^{d-i} A^{*h} E_d V.$$

Corollary 3.5 *With reference to Definition 2.1, let v_d denote a nonzero vector in $E_d V$ and consider the \mathbb{K} -linear transformation from \mathcal{D}^* to V which sends X to Xv_d for all $X \in \mathcal{D}^*$. Assume the equivalent conditions (i)–(iii) hold in Lemma 3.4. Then this linear transformation is an isomorphism.*

4 The existence of the split decomposition

We now display a necessary and sufficient condition for the existence of the split decomposition.

Theorem 4.1 *With reference to Definition 2.1, the following (i), (ii) are equivalent.*

(i) *There exists a decomposition of V which is split with respect to the orderings E_0, E_1, \dots, E_d and $E_0^*, E_1^*, \dots, E_d^*$.*

(ii) *Both*

$$E_i^* A E_j^* = \begin{cases} 0, & \text{if } i - j > 1; \\ \neq 0, & \text{if } i - j = 1 \end{cases} \quad (0 \leq i, j \leq d), \quad (7)$$

$$E_i A^* E_j = \begin{cases} 0, & \text{if } j - i > 1; \\ \neq 0, & \text{if } j - i = 1 \end{cases} \quad (0 \leq i, j \leq d). \quad (8)$$

Proof: (i) \Rightarrow (ii) By assumption there exists a decomposition U_0, U_1, \dots, U_d of V which is split with respect to the orderings E_0, E_1, \dots, E_d and $E_0^*, E_1^*, \dots, E_d^*$. For $0 \leq i \leq d$ we have $\sum_{h=0}^i U_h = \sum_{h=0}^i A^h E_0^* V$ by Lemma 2.4(i) and $\sum_{h=0}^i U_h = \sum_{h=0}^i E_h^* V$ by Lemma 2.4(ii) so $\sum_{h=0}^i E_h^* V = \sum_{h=0}^i A^h E_0^* V$. This gives Lemma 3.2(iii). Applying that lemma we obtain (7). For $0 \leq i \leq d$ we have $\sum_{h=i}^d U_h = \sum_{h=0}^{d-i} A^{*h} E_d V$ by Lemma 2.4(iii) and $\sum_{h=i}^d U_h = \sum_{h=i}^d E_h V$ by Lemma 2.4(iv) so $\sum_{h=i}^d E_h V = \sum_{h=0}^{d-i} A^{*h} E_d V$. This gives Lemma 3.4(iii). Applying that lemma we obtain (8).

(ii) \Rightarrow (i) For $0 \leq i \leq d$ we define $\tau_i = \prod_{h=0}^{i-1} (A - \theta_h I)$. We observe $\tau_0, \tau_1, \dots, \tau_d$ is a basis for the \mathbb{K} -vector space \mathcal{D} . Let v_0^* denote a nonzero vector in $E_0^* V$. Observe Lemma 3.2(i) holds by (7) so Corollary 3.3 applies; by that corollary $\tau_i v_0^*$ ($0 \leq i \leq d$) is a basis for V . We define $U_i = \text{Span}(\tau_i v_0^*)$ for $0 \leq i \leq d$ and observe U_0, U_1, \dots, U_d is a decomposition of V . We show this decomposition is split with respect to E_0, E_1, \dots, E_d and $E_0^*, E_1^*, \dots, E_d^*$. To do this we show the sequence U_0, U_1, \dots, U_d satisfies (1) and (2). Concerning (1), from the construction $(A - \theta_i I)\tau_i = \tau_{i+1}$ for $0 \leq i \leq d-1$ and $(A - \theta_d I)\tau_d = 0$. Applying both sides of these equations to v_0^* we find $(A - \theta_i I)U_i = U_{i+1}$ for $0 \leq i \leq d-1$ and $(A - \theta_d I)U_d = 0$. We have now shown (1). Concerning (2), this will follow if we can show (a) $(A^* - \theta_i^* I)U_i \subseteq \sum_{h=0}^{i-1} U_h$ for $0 \leq i \leq d$; (b) $(A^* - \theta_i^* I)U_i \subseteq \sum_{h=i-1}^d U_h$ for $1 \leq i \leq d$; (c) $(A^* - \theta_i^* I)U_i \neq 0$ for $1 \leq i \leq d$. We begin with (a). For $0 \leq j \leq d$ the elements $\{\tau_h | 0 \leq h \leq j\}$ and the elements $\{A^h | 0 \leq h \leq j\}$ span the same subspace of \mathcal{D} . Therefore $\sum_{h=0}^j U_h = \sum_{h=0}^j A^h E_0^* V$. We mentioned Lemma 3.2(i) holds so Lemma 3.2(iii) holds; therefore $\sum_{h=0}^j E_h^* V = \sum_{h=0}^j A^h E_0^* V$ so $\sum_{h=0}^j U_h = \sum_{h=0}^j E_h^* V$. Observe $(A^* - \theta_i^* I) \sum_{h=0}^i E_h^* V = \sum_{h=0}^{i-1} E_h^* V$ for $0 \leq i \leq d$. Combining these comments we find $(A^* - \theta_i^* I)U_i \subseteq \sum_{h=0}^{i-1} U_h$ for $0 \leq i \leq d$. We now have (a). Next we prove (b). From the construction, for $0 \leq j \leq d$ we have $\prod_{h=j}^d (A - \theta_h I)\tau_j = 0$ so $\prod_{h=j}^d (A - \theta_h I)U_j = 0$. From this we find $U_j \subseteq \sum_{h=j}^d E_h V$. Apparently $\sum_{h=i}^d U_h \subseteq \sum_{h=i}^d E_h V$ for $0 \leq i \leq d$. By this and since U_0, U_1, \dots, U_d is a decomposition we find $\sum_{h=i}^d U_h = \sum_{h=i}^d E_h V$ for $0 \leq i \leq d$. From (8) we find $A^* E_j V \subseteq \sum_{h=j-1}^d E_h V$ for $1 \leq j \leq d$. Therefore $(A^* - \theta_j^* I) \sum_{h=j}^d E_h V \subseteq \sum_{h=j-1}^d E_h V$ for $1 \leq j \leq d$. From these comments we find $(A^* - \theta_j^* I)U_j \subseteq \sum_{h=j-1}^d U_h$ for $1 \leq j \leq d$.

We now have (b). Next we show (c). Suppose there exists an integer i ($1 \leq i \leq d$) such that $(A^* - \theta_i^* I)U_i = 0$. We assume i is maximal subject to this. We obtain a contradiction as follows. For $i < j \leq d$ we find $(A^* - \theta_j^* I)U_j \subseteq U_{j-1}$ by (a), (b). In this inclusion the left-hand side is nonzero and the right-hand side has dimension 1 so we have equality. We mentioned earlier $(A - \theta_d I)U_d = 0$ so $U_d = E_d V$. Apparently $U_j = \prod_{h=j+1}^d (A^* - \theta_h^* I)E_d V$ for $i \leq j \leq d$. In particular $U_i = \prod_{h=i+1}^d (A^* - \theta_h^* I)E_d V$. Combining this with $(A^* - \theta_i^* I)U_i = 0$ we obtain $0 = \prod_{h=i}^d (A^* - \theta_h^* I)E_d V$. Let v_d denote a nonzero vector in $E_d V$ and observe $0 = \prod_{h=i}^d (A^* - \theta_h^* I)v_d$. This is inconsistent with Corollary 3.5 and the fact that $0 \neq \prod_{h=i}^d (A^* - \theta_h^* I)$. We now have a contradiction and (c) is proved. Combining (a)–(c) we obtain (2). We have shown the decomposition U_0, U_1, \dots, U_d satisfies (1), (2). Applying Definition 2.2 we find U_0, U_1, \dots, U_d is split with respect to the orderings E_0, E_1, \dots, E_d and $E_0^*, E_1^*, \dots, E_d^*$. \square

5 Two characterizations of a Leonard system

In this section we obtain two characterizations of a Leonard system, both of which involve the split decomposition. We will first state the characterizations, then prove a few lemmas, and then prove the characterizations. Our first characterization is stated as follows.

Theorem 5.1 *With reference to Definition 2.1, the sequence $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system if and only if both (i), (ii) hold below.*

- (i) *There exists a decomposition of V which is split with respect to the orderings E_0, E_1, \dots, E_d and $E_0^*, E_1^*, \dots, E_d^*$.*
- (ii) *There exists a decomposition of V which is split with respect to the orderings E_d, E_{d-1}, \dots, E_0 and $E_0^*, E_1^*, \dots, E_d^*$.*

In order to state our second characterization we recall a definition. Let $\sigma : \mathcal{A} \rightarrow \mathcal{A}$ denote any map. We call σ an *antiautomorphism* of \mathcal{A} whenever σ is an isomorphism of \mathbb{K} -vector spaces and $(XY)^\sigma = Y^\sigma X^\sigma$ for all $X, Y \in \mathcal{A}$. For example assume $\mathcal{A} = \text{Mat}_{d+1}(\mathbb{K})$. Then σ is an antiautomorphism of \mathcal{A} if and only if there exists an invertible $R \in \mathcal{A}$ such that $X^\sigma = R^{-1}X^t R$ for all $X \in \mathcal{A}$, where t denotes transpose. This follows from the Skolem-Noether Theorem [11, Cor. 9.122].

We now state our second characterization of a Leonard system.

Theorem 5.2 *With reference to Definition 2.1, the sequence $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system if and only if both (i), (ii) hold below.*

- (i) *There exists a decomposition of V which is split with respect to the orderings E_0, E_1, \dots, E_d and $E_0^*, E_1^*, \dots, E_d^*$.*
- (ii) *There exists an antiautomorphism \dagger of \mathcal{A} such that $A^\dagger = A$ and $A^{*\dagger} = A^*$.*

We now prove some lemmas which we will use to obtain Theorem 5.1 and Theorem 5.2. We have a preliminary remark. With reference to Definition 2.1, we consider the following four conditions:

$$E_i^* A E_j^* = \begin{cases} 0, & \text{if } i - j > 1; \\ \neq 0, & \text{if } i - j = 1 \end{cases} \quad (0 \leq i, j \leq d), \quad (9)$$

$$E_i^* A E_j^* = \begin{cases} 0, & \text{if } j - i > 1; \\ \neq 0, & \text{if } j - i = 1 \end{cases} \quad (0 \leq i, j \leq d), \quad (10)$$

$$E_i A^* E_j = \begin{cases} 0, & \text{if } i - j > 1; \\ \neq 0, & \text{if } i - j = 1 \end{cases} \quad (0 \leq i, j \leq d), \quad (11)$$

$$E_i A^* E_j = \begin{cases} 0, & \text{if } j - i > 1; \\ \neq 0, & \text{if } j - i = 1 \end{cases} \quad (0 \leq i, j \leq d). \quad (12)$$

We observe $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system if and only if each of (9)–(12) holds.

Lemma 5.3 *With reference to Definition 2.1, assume conditions (9) and (10) hold. Then A, E_0^* together generate \mathcal{A} . Moreover A, A^* together generate \mathcal{A} .*

Proof: Examining the proof of [15, Lemma 3.1] we find that the elements $A^r E_0^* A^s$ ($0 \leq r, s \leq d$) form a basis for the \mathbb{K} -vector space \mathcal{A} . It follows that A, E_0^* together generate \mathcal{A} . The elements A, A^* together generate \mathcal{A} since E_0^* is a polynomial in A^* . \square

Lemma 5.4 *With reference to Definition 2.1, assume conditions (9) and (10) hold. Then there exists a unique antiautomorphism \dagger of \mathcal{A} such that $A^\dagger = A$ and $A^{*\dagger} = A^*$. Moreover $X^{\dagger\dagger} = X$ for all $X \in \mathcal{A}$.*

Proof: Concerning the existence of \dagger , for $0 \leq i \leq d$ let v_i^* denote a nonzero element of $E_i^* V$ and recall $v_0^*, v_1^*, \dots, v_d^*$ is a basis for V . For $X \in \mathcal{A}$ let X^\flat denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ which represents X with respect to the basis $v_0^*, v_1^*, \dots, v_d^*$. We observe $\flat : \mathcal{A} \rightarrow \text{Mat}_{d+1}(\mathbb{K})$ is an isomorphism of \mathbb{K} -algebras. We abbreviate $B = A^\flat$ and $B^* = A^{*\flat}$. We observe B is irreducible tridiagonal and $B^* = \text{diag}(\theta_0^*, \theta_1^*, \dots, \theta_d^*)$. Let D denote the diagonal matrix in $\text{Mat}_{d+1}(\mathbb{K})$ which has ii entry

$$D_{ii} = \frac{B_{01} B_{12} \cdots B_{i-1,i}}{B_{10} B_{21} \cdots B_{i,i-1}} \quad (0 \leq i \leq d).$$

It is routine to verify $D^{-1} B^t D = B$. Each of D, B^* is diagonal so $DB^* = B^* D$; also $B^{*t} = B^*$ so $D^{-1} B^{*t} D = B^*$. Let $\sigma : \text{Mat}_{d+1}(\mathbb{K}) \rightarrow \text{Mat}_{d+1}(\mathbb{K})$ denote the map which satisfies $X^\sigma = D^{-1} X^t D$ for all $X \in \text{Mat}_{d+1}(\mathbb{K})$. We observe σ is an antiautomorphism of $\text{Mat}_{d+1}(\mathbb{K})$ such that $B^\sigma = B$ and $B^{*\sigma} = B^*$. We define the map $\dagger : \mathcal{A} \rightarrow \mathcal{A}$ to be the composition $\dagger := \flat \sigma \flat^{-1}$. We observe \dagger is an antiautomorphism of \mathcal{A} such that $A^\dagger = A$ and $A^{*\dagger} = A^*$. We have now shown there exists an antiautomorphism \dagger of \mathcal{A} such that $A^\dagger = A$ and $A^{*\dagger} = A^*$. This antiautomorphism is unique since A, A^* together generate \mathcal{A} . The map $X \rightarrow X^{\dagger\dagger}$ is an isomorphism of \mathbb{K} -algebras from \mathcal{A} to itself. This map is the identity since $A^{\dagger\dagger} = A$, $A^{*\dagger\dagger} = A^*$, and since A, A^* together generate \mathcal{A} . \square

Lemma 5.5 *With reference to Definition 2.1, assume there exists an antiautomorphism \dagger of \mathcal{A} such that $A^\dagger = A$ and $A^{*\dagger} = A^*$. Then $E_i^\dagger = E_i$ and $E_i^{*\dagger} = E_i^*$ for $0 \leq i \leq d$.*

Proof: Recall E_i (resp. E_i^*) is a polynomial in A (resp. A^*) for $0 \leq i \leq d$. \square

Lemma 5.6 *With reference to Definition 2.1, assume there exists an antiautomorphism \dagger of \mathcal{A} such that $A^\dagger = A$ and $A^{*\dagger} = A^*$. Then for $0 \leq i, j \leq d$, (i) $E_i^* A E_j^* = 0$ if and only if $E_j^* A E_i^* = 0$; and (ii) $E_i A^* E_j = 0$ if and only if $E_j A^* E_i = 0$.*

Proof: By Lemma 5.5 and since \dagger is an antiautomorphism,

$$(E_i^* A E_j^*)^\dagger = E_j^* A E_i^* \quad (0 \leq i, j \leq d).$$

Assertion (i) follows since $\dagger : \mathcal{A} \rightarrow \mathcal{A}$ is a bijection. To obtain (ii) interchange the roles of A and A^* in the proof of (i). \square

Lemma 5.7 *With reference to Definition 2.1, assume at least three of (9)–(12) hold. Then each of (9)–(12) hold; in other words $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system.*

Proof: Interchanging A and A^* if necessary, we may assume without loss of generality that (9) and (10) hold. By Lemma 5.4 there exists an antiautomorphism \dagger of \mathcal{A} such that $A^\dagger = A$ and $A^{*\dagger} = A^*$. By assumption at least one of (11), (12) holds. Combining this with Lemma 5.6 we find (11), (12) both hold. The result follows. \square

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1: By Theorem 4.1 we find (i) holds if and only if each of (9), (12) holds. Applying Theorem 4.1 again, this time with $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ replaced by $(A; A^*; \{E_{d-i}\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$, we find (ii) holds if and only if each of (9), (11) holds. Suppose $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system. Then each of (9)–(12) holds. In particular each of (9), (11), (12) holds so (i), (ii) hold by our above remarks. Conversely suppose (i), (ii) hold. Then each of (9), (11), (12) holds. At least three of (9)–(12) hold so $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system by Lemma 5.7. \square

We are now ready to prove Theorem 5.2.

Proof of Theorem 5.2: First assume $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system. Then (i) holds by Theorem 5.1 and (ii) holds by Lemma 5.4. Conversely assume (i), (ii) hold. Combining (i) and Theorem 4.1 we obtain (9), (12). Combining this with (ii) and using Lemma 5.6 we obtain (10), (11). Now each of (9)–(12) holds so $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system. \square

We would like to emphasize the following fact.

Theorem 5.8 *Let A, A^* denote a Leonard pair in \mathcal{A} . Then there exists a unique antiautomorphism \dagger of \mathcal{A} such that $A^\dagger = A$ and $A^{*\dagger} = A^*$. Moreover $X^{\dagger\dagger} = X$ for all $X \in \mathcal{A}$.*

Proof: Since A, A^* is a Leonard pair there exists an ordering E_0, E_1, \dots, E_d of the primitive idempotents of A and an ordering $E_0^*, E_1^*, \dots, E_d^*$ of the primitive idempotents of A^* such that $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system. These orderings satisfy (9)–(12). In particular (9), (10) are satisfied so the result follows by Lemma 5.4. \square

We finish this section with a comment.

Lemma 5.9 *With reference to Definition 2.1, assume there exists a decomposition of V which is split with respect to the orderings E_0, E_1, \dots, E_d and $E_0^*, E_1^*, \dots, E_d^*$. Then the following (i), (ii) are equivalent.*

(i) *The pair A, A^* is a Leonard pair.*

(ii) *The sequence $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system.*

Proof: (i) \Rightarrow (ii) We assume there exists a decomposition of V which is split with respect to the orderings E_0, E_1, \dots, E_d and $E_0^*, E_1^*, \dots, E_d^*$. Therefore each of (9), (12) holds by Theorem 4.1. Since A, A^* is a Leonard pair there exists an antiautomorphism \dagger of \mathcal{A} such that $A^\dagger = A$ and $A^{*\dagger} = A^*$. Applying Lemma 5.6 we find each of (10), (11) holds. Now each of (9)–(12) holds so $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system.

(ii) \Rightarrow (i) Clear. \square

6 The two characterizations in terms of matrices

In this section we restate Theorem 5.1 and Theorem 5.2 in terms of matrices. We first set some notation. With reference to Definition 2.1, suppose there exists a decomposition U_0, U_1, \dots, U_d of V which is split with respect to the orderings E_0, E_1, \dots, E_d and $E_0^*, E_1^*, \dots, E_d^*$. Pick an integer i ($1 \leq i \leq d$). By (2) we find $(A^* - \theta_i^* I)U_i = U_{i-1}$ and by (1) we find $(A - \theta_{i-1} I)U_{i-1} = U_i$. Apparently U_i is an eigenspace for $(A - \theta_{i-1} I)(A^* - \theta_i^* I)$ and the corresponding eigenvalue is a nonzero element of \mathbb{K} . Let us denote this eigenvalue by φ_i . We call $\varphi_1, \varphi_2, \dots, \varphi_d$ the *split sequence* for A, A^* with respect to the orderings E_0, E_1, \dots, E_d and $E_0^*, E_1^*, \dots, E_d^*$. The split sequence has the following interpretation. For $0 \leq i \leq d$ let u_i denote a nonzero vector in U_i and recall u_0, u_1, \dots, u_d is a basis for V . We normalize the u_i so that $(A - \theta_i I)u_i = u_{i+1}$ for $0 \leq i \leq d-1$. With respect to the basis u_0, u_1, \dots, u_d the matrices which represent A and A^* are as follows.

$$A : \begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & 1 & \theta_d \end{pmatrix}, \quad A^* : \begin{pmatrix} \theta_0^* & \varphi_1 & & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & & \\ & & \theta_2^* & \cdot & \\ & & & \ddots & \cdot \\ \mathbf{0} & & & & \varphi_d \\ & & & & \theta_d^* \end{pmatrix}.$$

Motivated by this we consider the following set-up.

Definition 6.1 Let d denote a nonnegative integer. Let A and A^* denote matrices in $\text{Mat}_{d+1}(\mathbb{K})$ of the form

$$A = \begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ & 1 & \theta_1 & & \\ & & 1 & \theta_2 & \\ & & & \ddots & \ddots \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \quad A^* = \begin{pmatrix} \theta_0^* & \varphi_1 & & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & & \\ & & \theta_2^* & \ddots & \\ & & & \ddots & \ddots \\ \mathbf{0} & & & & \varphi_d & \theta_d^* \end{pmatrix},$$

where

$$\begin{aligned} \theta_i &\neq \theta_j, & \theta_i^* &\neq \theta_j^* & \text{ if } i &\neq j & (0 \leq i, j \leq d), \\ \varphi_i &\neq 0 & & & (1 \leq i \leq d). \end{aligned}$$

We observe A (resp. A^*) is multiplicity-free, with eigenvalues $\theta_0, \theta_1, \dots, \theta_d$ (resp. $\theta_0^*, \theta_1^*, \dots, \theta_d^*$.) For $0 \leq i \leq d$ we let E_i (resp. E_i^*) denote the primitive idempotent for A (resp. A^*) associated with θ_i (resp. θ_i^* .)

We have some comments. With reference to Definition 6.1, for $0 \leq i \leq d$ let u_i denote the vector in \mathbb{K}^{d+1} which has i th entry 1 and all other entries 0. We observe u_0, u_1, \dots, u_d is a basis for \mathbb{K}^{d+1} . From the form of A we have $(A - \theta_i I)u_i = u_{i+1}$ for $0 \leq i \leq d-1$ and $(A - \theta_d I)u_d = 0$. From the form of A^* we have $(A^* - \theta_i^* I)u_i = \varphi_i u_{i-1}$ for $1 \leq i \leq d$ and $(A^* - \theta_0^* I)u_0 = 0$. For $0 \leq i \leq d$ let U_i denote the subspace of \mathbb{K}^{d+1} spanned by u_i . Then U_0, U_1, \dots, U_d is a decomposition of \mathbb{K}^{d+1} . This decomposition satisfies $(A - \theta_i I)U_i = U_{i+1}$ for $0 \leq i \leq d-1$ and $(A - \theta_d I)U_d = 0$. Similarly $(A^* - \theta_i^* I)U_i = U_{i-1}$ for $1 \leq i \leq d$ and $(A^* - \theta_0^* I)U_0 = 0$. In other words the decomposition U_0, U_1, \dots, U_d is split with respect to the orderings E_0, E_1, \dots, E_d and $E_0^*, E_1^*, \dots, E_d^*$. We observe $\varphi_1, \varphi_2, \dots, \varphi_d$ is the corresponding split sequence for A, A^* . We now consider when is the pair A, A^* a Leonard pair. We begin with a remark.

Lemma 6.2 With reference to Definition 6.1, the following (i), (ii) are equivalent.

- (i) The pair A, A^* is a Leonard pair.
- (ii) The sequence $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system.

Proof: We mentioned there exists a decomposition of \mathbb{K}^{d+1} which is split with respect to the orderings E_0, E_1, \dots, E_d and $E_0^*, E_1^*, \dots, E_d^*$. Therefore Lemma 5.9 applies and the result follows. \square

We now give a matrix version of Theorem 5.1.

Theorem 6.3 Referring to Definition 6.1, the following (i), (ii) are equivalent.

- (i) The pair A, A^* is a Leonard pair.

(ii) There exists an invertible $G \in \text{Mat}_{d+1}(\mathbb{K})$ and there exists nonzero $\phi_i \in \mathbb{K}$ ($1 \leq i \leq d$) such that

$$G^{-1}AG = \begin{pmatrix} \theta_d & & & & \mathbf{0} \\ 1 & \theta_{d-1} & & & \\ & 1 & \theta_{d-2} & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & 1 & \theta_0 \end{pmatrix}, \quad G^{-1}A^*G = \begin{pmatrix} \theta_0^* & \phi_1 & & & \mathbf{0} \\ & \theta_1^* & \phi_2 & & \\ & & \theta_2^* & \ddots & \\ & & & \ddots & \phi_d \\ \mathbf{0} & & & & \theta_d^* \end{pmatrix}.$$

Suppose (i), (ii) hold. Then the sequence $\phi_1, \phi_2, \dots, \phi_d$ is the split sequence for A, A^* associated with the orderings E_d, E_{d-1}, \dots, E_0 and $E_0^*, E_1^*, \dots, E_d^*$.

Proof: (i) \Rightarrow (ii) The sequence $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system by Lemma 6.2. By Theorem 5.1 there exists a decomposition of \mathbb{K}^{d+1} which is split with respect to the orderings E_d, E_{d-1}, \dots, E_0 and $E_0^*, E_1^*, \dots, E_d^*$. Let V_0, V_1, \dots, V_d denote this decomposition. By the definition of a split decomposition we have $(A - \theta_{d-i}I)V_i = V_{i+1}$ for $0 \leq i \leq d-1$ and $(A - \theta_0I)V_d = 0$. Moreover $(A^* - \theta_i^*I)V_i = V_{i-1}$ for $1 \leq i \leq d$ and $(A^* - \theta_0^*I)V_0 = 0$. For $0 \leq i \leq d$ let v_i denote a nonzero vector in V_i and observe v_0, v_1, \dots, v_d is a basis for \mathbb{K}^{d+1} . We normalize the v_i so that $(A - \theta_{d-i}I)v_i = v_{i+1}$ for $0 \leq i \leq d-1$. Let $\phi_1, \phi_2, \dots, \phi_d$ denote the split sequence for A, A^* with respect to the orderings E_d, E_{d-1}, \dots, E_0 and $E_0^*, E_1^*, \dots, E_d^*$. Then $\phi_i \neq 0$ ($1 \leq i \leq d$) and moreover $(A^* - \theta_i^*I)v_i = \phi_i v_{i-1}$ ($1 \leq i \leq d$), $(A^* - \theta_0^*I)v_0 = 0$. Let G denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ which has column i equal to v_i for $0 \leq i \leq d$. We observe G is invertible. Moreover the matrices $G^{-1}AG$ and $G^{-1}A^*G$ have the form shown above.

(ii) \Rightarrow (i) We show $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system. In order to do this we apply Theorem 5.1. In the paragraph after Definition 6.1 we mentioned there exists a decomposition of \mathbb{K}^{d+1} which is split with respect to the orderings E_0, E_1, \dots, E_d and $E_0^*, E_1^*, \dots, E_d^*$. Therefore Theorem 5.1(i) holds. We show Theorem 5.1(ii) holds. For $0 \leq i \leq d$ let v_i denote column i of G and observe v_0, v_1, \dots, v_d is a basis for \mathbb{K}^{d+1} . From the form of $G^{-1}AG$ we find $(A - \theta_{d-i}I)v_i = v_{i+1}$ for $0 \leq i \leq d-1$ and $(A - \theta_0I)v_d = 0$. From the form of $G^{-1}A^*G$ we find $(A^* - \theta_i^*I)v_i = \phi_i v_{i-1}$ for $1 \leq i \leq d$ and $(A^* - \theta_0^*I)v_0 = 0$. For $0 \leq i \leq d$ let V_i denote the subspace of \mathbb{K}^{d+1} spanned by v_i . Then V_0, V_1, \dots, V_d is a decomposition of \mathbb{K}^{d+1} . Also $(A - \theta_{d-i}I)V_i = V_{i+1}$ for $0 \leq i \leq d-1$ and $(A - \theta_0I)V_d = 0$. Moreover $(A^* - \theta_i^*I)V_i = V_{i-1}$ for $1 \leq i \leq d$ and $(A^* - \theta_0^*I)V_0 = 0$. Apparently V_0, V_1, \dots, V_d is split with respect to the orderings E_d, E_{d-1}, \dots, E_0 and $E_0^*, E_1^*, \dots, E_d^*$. Now Theorem 5.1(ii) holds; applying that theorem we find $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system. In particular A, A^* is a Leonard pair. Assume (i), (ii) both hold. From the proof of (ii) \Rightarrow (i) we find that for $1 \leq i \leq d$, ϕ_i is the eigenvalue of $(A - \theta_{d-i+1}I)(A^* - \theta_i^*I)$ associated with V_i . Therefore $\phi_1, \phi_2, \dots, \phi_d$ is the split sequence for A, A^* associated with the orderings E_d, E_{d-1}, \dots, E_0 and $E_0^*, E_1^*, \dots, E_d^*$. \square

We now give a matrix version of Theorem 5.2.

Theorem 6.4 Referring to Definition 6.1, the following (i), (ii) are equivalent.

(i) The pair A, A^* is a Leonard pair.

(ii) There exists an invertible $H \in \text{Mat}_{d+1}(\mathbb{K})$ such that

$$H^{-1}A^tH = A, \quad H^{-1}A^{*t}H = A^*.$$

Proof: (i) \Rightarrow (ii) By Theorem 5.8 there exists an antiautomorphism \dagger of $\text{Mat}_{d+1}(\mathbb{K})$ such that $A^\dagger = A$ and $A^{*\dagger} = A^*$. Since \dagger is an antiautomorphism there exists an invertible $H \in \text{Mat}_{d+1}(\mathbb{K})$ such that $X^\dagger = H^{-1}X^tH$ for all $X \in \text{Mat}_{d+1}(\mathbb{K})$. Setting $X = A$ we have $H^{-1}A^tH = A$. Setting $X = A^*$ we have $H^{-1}A^{*t}H = A^*$.

(ii) \Rightarrow (i) We show $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system. In order to do this we apply Theorem 5.2. In the paragraph after Definition 6.1 we mentioned there exists a decomposition of \mathbb{K}^{d+1} which is split with respect to the orderings E_0, E_1, \dots, E_d and $E_0^*, E_1^*, \dots, E_d^*$. Therefore Theorem 5.2(i) holds. Let $\dagger : \text{Mat}_{d+1}(\mathbb{K}) \rightarrow \text{Mat}_{d+1}(\mathbb{K})$ denote the map which satisfies $X^\dagger = H^{-1}X^tH$ for all $X \in \text{Mat}_{d+1}(\mathbb{K})$. Then \dagger is an antiautomorphism of $\text{Mat}_{d+1}(\mathbb{K})$ such that $A^\dagger = A$ and $A^{*\dagger} = A^*$. Now Theorem 5.2(ii) holds; applying that theorem we find $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system. In particular A, A^* is a Leonard pair. \square

7 Remarks

Referring to Definition 6.1, presumably condition (ii) of Theorem 6.3 or Theorem 6.4 can be translated into a condition on the entries of A and A^* . We obtained such a condition in [13]; we cite it here for the sake of completeness.

Theorem 7.1 [13, Corollary 14.2] *With reference to Definition 6.1, the pair A, A^* is a Leonard pair if and only if there exists nonzero $\phi_i \in \mathbb{K}$ ($1 \leq i \leq d$) such that (i)–(iii) hold below.*

$$(i) \quad \varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d) \quad (1 \leq i \leq d).$$

$$(ii) \quad \phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0) \quad (1 \leq i \leq d).$$

(iii) The expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i for $2 \leq i \leq d-1$.

Suppose (i)–(iii) hold. Then $\phi_1, \phi_2, \dots, \phi_d$ is the split sequence for A, A^* with respect to the orderings E_d, E_{d-1}, \dots, E_0 and $E_0^*, E_1^*, \dots, E_d^*$.

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